

Contents

| | |
|----------------------------------------------------------------------|-----|
| Preface | vii |
| 1 Introduction to the Ramanujan conjectures | 1 |
| 1.1. The τ -function | 1 |
| 1.2. The multiplicativity of $\tau(n)$ | 4 |
| 1.3. Hecke theory | 10 |
| 2 Modular Forms and Dirichlet Series | 13 |
| 2.1. L -series attached to modular forms | 13 |
| 2.2. Motivation for the study of symmetric power L -functions | 17 |
| 3 The Rankin-Selberg method | 19 |
| 3.1. Rankin's approach to the Ramanujan conjecture | 19 |
| 3.2. Eisenstein series | 23 |
| 3.3. A general theorem | 28 |
| 4 Relation to the Weil conjectures | 31 |
| 4.1. Artin's thesis | 31 |
| 4.2. The Weil conjectures | 34 |
| 4.3. The Kuga-Sato variety | 36 |
| 5 The maximal order of $\tau(n)$ | 37 |
| 5.1. Symmetric power L -functions | 37 |
| 6 The Riemann and Hurwitz zeta functions | 41 |
| 6.1. The Riemann zeta function | 41 |
| 6.2. Dirichlet L -series | 43 |
| 6.3. The Hurwitz zeta function | 43 |
| 6.4. Analytic continuation of the Riemann and Hurwitz zeta functions | 44 |
| 6.5. The location of poles | 45 |
| 6.6. Lerch's formula | 46 |

| | |
|--------------------------------------------------------------------|-----|
| 6.7. Special values of the Hurwitz zeta function | 47 |
| 6.8. Special values of Dirichlet L -series | 47 |
| 6.9. Ramanujan's formula for $\zeta(2k + 1)$ | 50 |
| 7 The prime number theorem and generalizations | 53 |
| 7.1. The Dedekind zeta function | 53 |
| 7.2. Landau's Theorem | 54 |
| 7.3. The Wiener-Ikehara Tauberian theorem | 56 |
| 7.4. The proof of the Tauberian theorem | 60 |
| 7.5. A non-vanishing theorem | 62 |
| 7.6. Dirichlet hyperbola method | 63 |
| 7.7. A more general identity | 66 |
| 8 Hecke L-series | 69 |
| 8.1. Ideal class zeta functions | 69 |
| 8.2. The prime ideal theorem | 72 |
| 9 Artin L-functions | 77 |
| 9.1. Analytic continuation and non-vanishing of L -series | 77 |
| 9.2. Non-abelian L -functions | 78 |
| 9.3. The Chebotarev density theorem | 80 |
| 9.4. Zeros and poles of Artin L -series | 82 |
| 10 Selberg's conjectures and Artin L-functions | 85 |
| 10.1. Langlands reciprocity conjecture | 85 |
| 10.2. Selberg's conjectures | 88 |
| 11 More on the Sato-Tate conjecture | 93 |
| 11.1. The Tauberian theorem and equidistribution | 93 |
| 11.2. The Rankin-Selberg L -function | 94 |
| 11.3. Symmetric power representations | 94 |
| 11.4. An outline of the proof | 96 |
| 12 Automorphy and the Sato-Tate conjecture | 99 |
| 12.1. Introduction | 99 |
| 12.2. Preliminary results on L -functions | 102 |
| 12.3. Preliminary results on equidistribution | 105 |

| | |
|---------------------------------------------------------------|-----|
| 12.4. Functional equations for symmetric power L -functions | 109 |
| 12.5. Proof of Theorem 24 | 110 |
| 12.6. Effective error estimates and proof of Theorem 27 | 111 |
| 12.7. Concluding remarks | 112 |
| References | 115 |